Robust Resistor Network Topology Design by Conic Optimization

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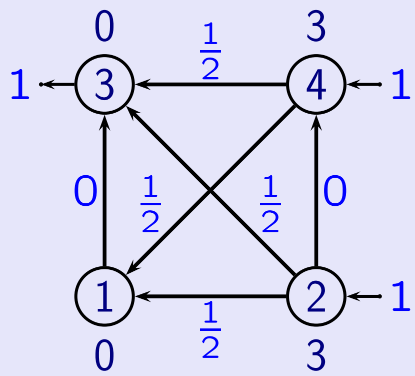
The resistance network topology design (RNTD) problem

A resistance network consists of resistances linked to each other at the nodes of the network. Some nodes are connected to earth, these are called fixed nodes; the remaining nodes are free nodes.

By way of example, consider a network with 4 nodes, numbered from 1 to 4, with a resistor of 6 Ohm between every pair of nodes. We take node 1 fixed and the external currents in the nodes 2, 3 and 4 are 1, -1 and 1 respectively. Then the currents and potentials are as indicated.



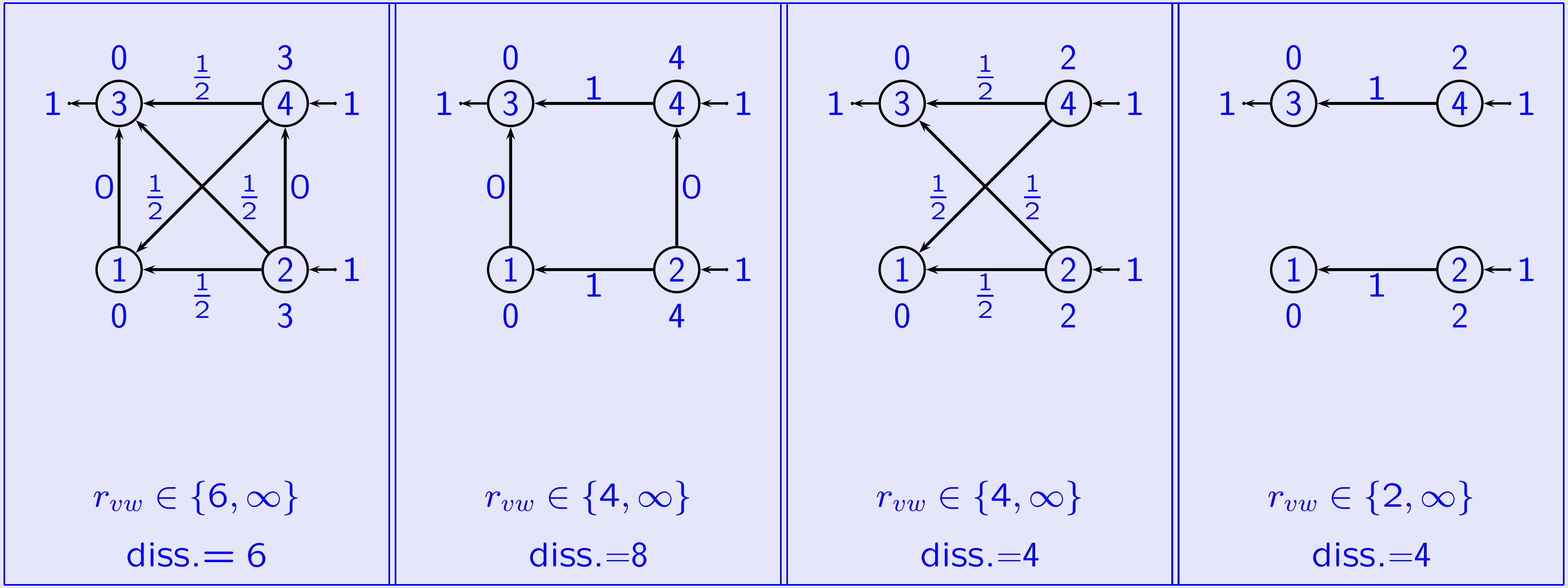
The power consumption, or dissipation in the network is given by

$$4 \times \left(\frac{1}{2}\right)^2 \times 6 = 6.$$

Given the external currents at the free nodes, our goal is to design a network with minimal dissipation, assuming that the sum of conductance values is fixed. This is the RNTD problem.

Other solutions for the sample problem

In the sample problem the sum of the conductance values is $6 \times \frac{1}{6} = 1$. The figure below shows that there exist other networks on 4 nodes, with the same property, and with different dissipations.



As becomes clear later on, the last two networks have the smallest possible dissipation for the given external currents.

Balance equations provided by Kirchhoff's first law

Let V denote the set of free nodes in the network. For each node $v \in V$, y_v will denote the potential of v (if v is a fixed node then its potential is zero).

To each resistor we associate a directed arc (v, w) connecting the corresponding end nodes of the resistor. The set of all arcs is denoted as $\mathcal{A}$. The structural matrix is the $|V| \times |\mathcal{A}|$ matrix B defined by

$$B(u,(v,w)) = \begin{cases} 1, & \text{if } u = v, \\ -1, & \text{if } u = w, \\ 0, & \text{otherwise.} \end{cases} \qquad u \in V,\ (v,w) \in \mathcal{A}.$$

The external current is denoted as a vector $f \in \mathbf{R}^V$. For each $v \in V$, f_v may be either positive, negative or zero.

The currents on the arcs in the network are denoted as x_{vw}, and x is the vector of all currents. So $x \in \mathbf{R}^{\mathcal{A}}$. Kirchhoff's first law gives the balance equations in the free nodes, which are simply

$$Bx = f.$$

Equations provided by Ohm's law

We denote by y the vector of the potentials y_v in the free nodes. By Ohm's law, on each arc $(v,w) \in \mathcal{A}$ we must have $x_{vw} r_{vw} = y_v - y_w$, where r_{vw} denote the resistance value of the arc (v,w). In other words,

$$B^T y = Rx,$$

where $R = \mathrm{diag}\,(r)$.

If $r_{vw} = 0$ then we may identify the nodes v and w. So we assume, without loss of generality, $r_{vw} > 0$ for all arcs (v,w). Resistance values may be ∞ (if the corresponding nodes are not connected). The conductance value of arc (v,w) is denoted as g_{vw}. So $g_{vw} = 1/r_{vw}$, for each arc. Defining $G = R^{-1}$, the last equation is equivalent to

$$x = GB^T y.$$

Substitution into $Bx = f$ yields

$$BGB^T y = f.$$

The matrix BGB^T is called the conductance matrix of the network and denoted $A(g)$. Note that $A(g)$ is a $|V| \times |V|$ matrix which depends linearly on the conductance values.

Dissipation

To complete our model, we must express the dissipation of the network as a function of the external current vector f and the conductance vector g. The dissipation is given by

$$\mathrm{Diss}_f(g) = \sum_{\{v,w\}\in\mathcal{A}} x_{vw}^2 R_{vw} = x^T R x.$$

Using

$$\begin{aligned} B^T y &= Rx \\ Bx &= f \\ BGB^T y &= f, \end{aligned}$$

this can be reduced as follows:

$$\mathrm{Diss}_f(g) = x^T R x = x^T \left(B^T y\right) = (Bx)^T y = f^T y = y^T B G B^T y.$$

Since the (v, w)-entry of $B^T y$ equals $y_v - y_w$ this implies

$$\mathrm{Diss}_f(g) = \left(B^T y\right)^T R^{-1} B^T y = \sum_{\{v,w\}\in\mathcal{A}} \frac{|y_v - y_w|^2}{r_{vw}},$$

as it should.

Illustration: Replacement resistance

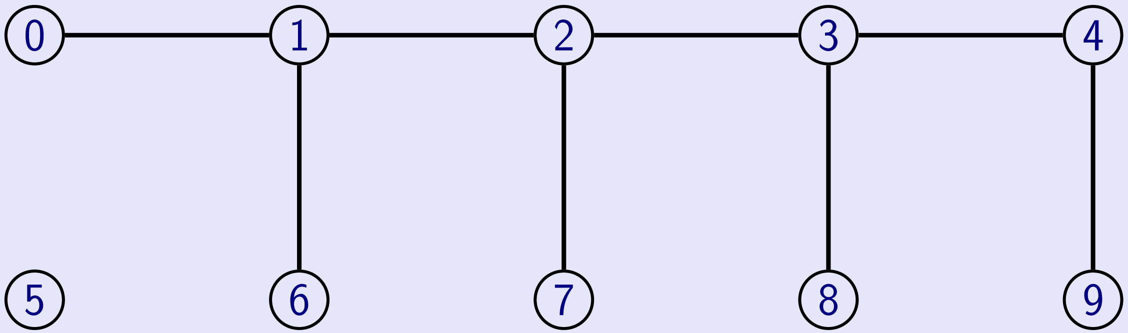
When entering a unit current e_v at a free node v, the replacement resistance r_v at this node (to earth) satisfies $y_v = r_v \cdot 1 = r_v$. So r_v to is equal to the potential y_v at the same node. Due to $BGB^T y = f$, it follows that $y = \left(BGB^T\right)^{-1} f = \left(BGB^T\right)^{-1} e_v$, whence

$$r_v = y_v = e_v^T \left(BGB^T\right)^{-1} e_v = A(g)^{-1}_{vv}.$$

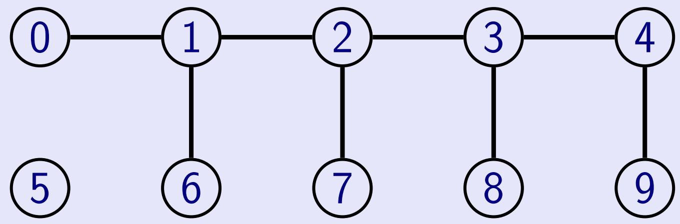
In words:

> the replacement resistance at any free node is equal to the corresponding element on the diagonal of the inverse matrix of the conductance matrix $A(g)$.

As an example we consider the network drawn below.



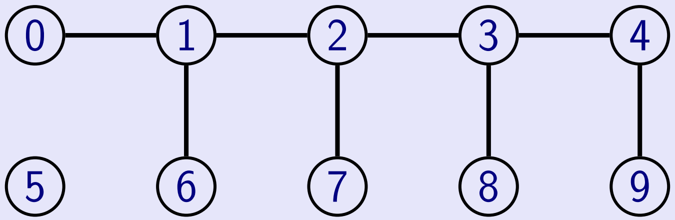
The nodes are numbered as indicated and the resistance values at the drawn arcs all are equal to 1. The nodes 5 to 9 are taken fixed.



The matrix B is as follows:

$$B = \begin{array}{cc} & \begin{array}{cccccccc} (0,1) & (1,2) & (2,3) & (3,4) & (1,6) & (2,7) & (3,8) & (4,9) \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & & & & & & & \\ -1 & 1 & & & 1 & & & \\ & -1 & 1 & & & 1 & & \\ & & -1 & 1 & & & 1 & \\ & & & -1 & & & & 1 \end{bmatrix} \end{array}.$$

Only the nonzero entries of B are indicated.



Since R is the unit matrix I, also $G = I$ and hence the conductance matrix satisfies $A(g) = BGB^T = BB^T$:

$$A(g) = \begin{bmatrix} 1 & -1 & & & \\ -1 & 3 & -1 & & \\ & -1 & 3 & -1 & \\ & & -1 & 3 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad A(g)^{-1} = \frac{1}{21}\begin{bmatrix} 34 & 13 & 5 & 2 & 1 \\ 13 & 13 & 5 & 2 & 1 \\ 5 & 5 & 10 & 4 & 2 \\ 2 & 2 & 4 & 10 & 5 \\ 1 & 1 & 2 & 5 & 13 \end{bmatrix}.$$

We conclude, e.g., that the replacement resistance at node 0, i.e., the resistance from node 0 to earth, equals $\frac{34}{21}$. Similarly, at nodes 1 and 4 the replacement resistance is $\frac{13}{21}$, and at nodes 2 and 3 the replacement resistance is $\frac{10}{21}$.

Nonlinear model for the RNTD-problem

Given the nodes, the fixed nodes and the external current f, we want to find conductance values such that the dissipation is minimized. This amounts to solving the following minimization problem:

$$\min_{g,y} \left\{ f^T y \;:\; A(g)y = f,\, g \geq 0 \right\}.$$

If y and g are feasible, and $\lambda > 0$, then g/λ and λy are feasible as well. Hence, letting λ decrease to zero the objective value approaches to zero. But then all nonzero conductance values go to infinity. To prevent this, we put an upper bound on the sum of the conductances:

$$\min_{g,y} \left\{ f^T y \;:\; A(g)y = f,\, g \geq 0,\, \sum_{\{v,w\}\in\mathcal{A}} g_{vw} \leq w \right\}.$$

Since $A(g)$ is linear in g, the equality constraints in the problem are nonlinear (not convex). As a consequence, in this form the problem cannot be solved efficiently.

To a linear model for the RNTD-problem

$$(NLO) \qquad \min_{g,y} \left\{ f^T y \; : \; A(g)y = f,\, e^T g \le w,\, g \ge 0 \right\}.$$

Define $\Delta = \{g \; : \; e^T g \le w,\, g \ge 0\}$, and assume that $A(g)$ is invertible. By eliminating y we obtain a problem in the variable vector g only:

$$\min_{g \in \Delta} \; f^T A(g)^{-1} f.$$

Yet we observe that

$$f^T A(g)^{-1} f = \max_y \left(\underbrace{2f^T y - y^T A(g) y}_{E_{g,f}(y)} \right),$$

and that each maximizer of the concave function $E_{g,f}(y)$ satisfies $A(g)y = f$. Hence the problem (NLO) can be restated as

$$\min_{g \in \Delta} \max_y \; E_{g,f}(y) \equiv \max_y \; \min_{g \in \Delta} \; E_{g,f}(y).$$

Taking y fixed, and using that $y^T A(g) y = y^T B G B^T y = \left(B^T y\right)^T G B^T y$, we obtain

$$\min_{g \in \Delta} \; E_{g,f}(y) = 2f^T y - \max_{g \in \Delta} \sum_{vw \in \mathcal{A}} g_{vw} (B^T y)^2_{vw} \ge 2f^T y - \sum_{vw \in \mathcal{A}} g_{vw} \left\| B^T y \right\|^2_\infty = 2f^T y - w \left\| B^T y \right\|^2_\infty.$$

Since $(B^T y)_{vw} = y_v - y_w$, the last equality holds with equality if and only if

$$e^T g = w \quad \text{and} \quad g_{vw} > 0 \Rightarrow |y_v - y_w| = \left\| B^T y \right\|_\infty = \max_{vw \in \mathcal{A}} |y_v - y_w|.$$

We conclude that

a network g is optimal if and only if $e^T g = w$ and all resistors connect nodes for which the (absolute) voltages are equal and maximal among all voltage values.

Thus we have reduced (NLO) to the following maximization problem in y:

$$\max_{y}\ \left(2f^T y - w\left\|B^T y\right\|_\infty^2\right) = \max_{y,\,\lambda}\ \left(2\lambda f^T y - w\lambda^2\left\|B^T y\right\|_\infty^2\right) = \max_{y}\ \frac{\left(f^T y\right)^2}{w\left\|B^T y\right\|_\infty^2}.^*$$

The last expression is homogeneous in y. Since $f^T y \geq 0$, this problem can be solved by fixing $\|B^T y\|_\infty$ and maximizing $f^T y$, which is essentially a linear problem:

$$(LOP) \qquad \max_{y}\left\{f^T y\ :\ \left\|B^T y\right\|_\infty = 1\right\} = \max_{y}\left\{f^T y\ :\ \left\|B^T y\right\|_\infty \leq 1\right\}.$$

An optimal solution y^* for this linear problem is, up to a scalar factor, μ say, optimal for (NLO). Thus (NLO) reduces to a problem in μ whose optimal value is known:

$$\min_{g,\,\mu}\left\{f^T(\mu y^*)\ :\ A(g)(\mu y^*) = f,\, e^T g = w,\, g \geq 0\right\} = \frac{\left(f^T y^*\right)^2}{w}.$$

From this we deduce that $\mu = \frac{f^T y^*}{w}$ and that g can be found by solving the linear system

$$A(g)y^* = \frac{wf}{f^T y^*}, \quad e^T g = w,\, g \geq 0, \quad |y_v^* - y_w^*| < \left\|B^T y^*\right\|_\infty \Rightarrow g_{vw} = 0.$$

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*Here we use that, when fixing y, the best value for λ satisfies $2f^T y - 2w\lambda\|B^T y\|_\infty^2 = 0$.

Application to the first example

The matrix B, the vector f and w are as follows.

$$B = \begin{array}{c} \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} (2,1) & (1,3) & (4,1) & (2,3) & (2,4) & (4,3) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \end{array}, \quad f = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

The optimal solution y^* of the linear problem is such that

$$y^* = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad B^T y^* = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad f^T y^* = 2.$$

Since $w = 1$, the minimal dissipation is 4 and this is achieved if $g_{13} = g_{24} = 0$ and

$$\begin{bmatrix} g_{21} & 0 & g_{23} & 0 \\ 0 & 0 & -g_{23} & -g_{43} \\ 0 & g_{41} & 0 & g_{43} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad g_{21} + g_{41} + g_{23} + g_{43} = 1.$$

One may easily deduce from this that the last two solutions on sheet 4 are indeed optimal.

Instability with respect to small perturbations

Consider a network g which is optimal for the external (design) current f. We now demonstrate that small perturbations in the design current f may have a disastrous effect on the dissipation.

Let u be an eigenvector of $A(g)$ with eigenvalue λ. So $A(g)u = \lambda u$. Without loss of generality we assume $\|u\| = \|f\|$ and $u^T f \geq 0$. Consider the situation that the external current is perturbed as follows:

$$f(\gamma) = f + \gamma u$$

for some $\gamma \geq 0$. Then the potential vector $y(\gamma)$ under the new current follows by solving the equation $A(g)y(\gamma) = f + \gamma u$, which gives

$$y(\gamma) = y + \frac{\gamma}{\lambda} u,$$

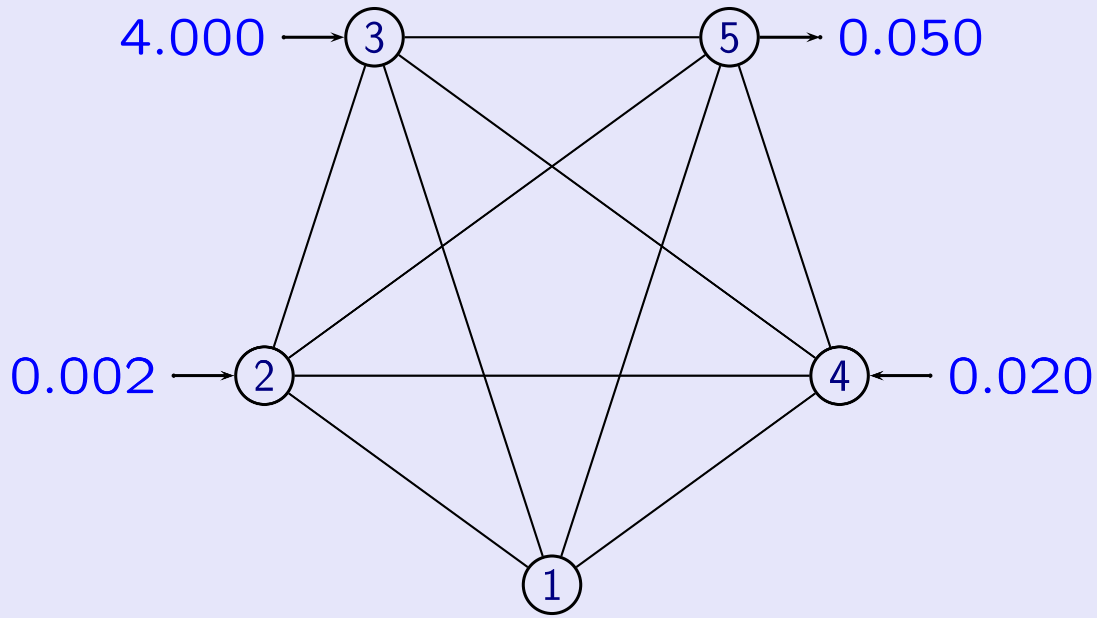
where y denotes the potential vector with respect to f. Here we used $A(g)y = f$ and $A(g)u = \lambda u$. The new dissipation then satisfies

$$\mathsf{Diss}_{f(\gamma)}(g) = (f + \gamma u)^T \left(y + \frac{\gamma}{\lambda} u \right) = f^T y + \gamma u^T y + \frac{\gamma}{\lambda} f^T u + \frac{\gamma^2}{\lambda} u^T u \geq \mathsf{Diss}_f(g) + \frac{\gamma^2}{\lambda} \|f\|^2.$$

Here we used that $\mathsf{Diss}_f(g) = f^T y$, $f^T u \geq 0$ (and hence also $u^T y \geq 0$) and $u^T u = \|f\|^2$. We conclude that the effect on the dissipation of a small perturbation of the design current may be large if the eigenvalue λ is small.

Example of an instable network

By way of example we consider the 5-node grid shown below. Node 1 is fixed and the values of the input currents in the remaining nodes are as shown.



Example of an instable network (cont.)

The matrix B and the vector f are as follows.

$$
B = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} (1,2) & (1,3) & (1,4) & (1,5) & (2,3) & (2,4) & (2,5) & (3,4) & (3,5) & (4,5) \end{array} \\
\left[\begin{array}{cccccccccc}
-1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1
\end{array}\right]
\end{array}, \quad
f = \begin{bmatrix} 0.002 \\ 4.000 \\ 0.020 \\ -0.050 \end{bmatrix}.
$$

An optimal solution is given by:

$$
y = \begin{bmatrix} 4.022282 \\ 4.022000 \\ 4.021954 \\ -0.000004 \end{bmatrix}, \quad
g = \begin{bmatrix} 0.000265 \\ 0.984046 \\ 0.003257 \\ 0 \\ 0 \\ 0 \\ 0.000232 \\ 0 \\ 0.010484 \\ 0.001715 \end{bmatrix}, \quad
x = \begin{bmatrix} -0.001066 \\ -3.957833 \\ -0.013100 \\ 0 \\ 0 \\ 0 \\ 0.000934 \\ 0 \\ 0.042167 \\ 0.006900 \end{bmatrix}.
$$

The dissipation is 16.176484.

Example of an instable network (cont.)

The matrix $A(g)$ is given by

$$A(g) = B\,\mathrm{diag}\,(g)B^T = \begin{bmatrix} 0.000497 & 0 & 0 & -0.000232 \\ 0 & 0.994530 & 0 & -0.010484 \\ 0 & 0 & 0.004973 & -0.001715 \\ -0.000233 & -0.010484 & -0.001715 & 0.012432 \end{bmatrix}.$$

Its eigenvalues are 0.000492, 0.004593, 0.012705, 0.994642. The smallest eigenvalue is $\lambda = 0.000492$, and the corresponding eigenvector u with $\|u\| = \|f\|$, $f^T u \geq 0$ is

$$u = \begin{bmatrix} 3.999373 \\ 0.000877 \\ 0.031823 \\ 0.083111 \end{bmatrix}.$$

With $\gamma = 0.1$ we obtain

$$f_2 = f(0.1) = f + 0.1u = \begin{bmatrix} 0.401937 \\ 4.000088 \\ 0.023182 \\ -0.041689 \end{bmatrix}.$$

Example of an instable network (cont.)

$$\lambda = 0.000492, \qquad u = \begin{bmatrix} 3.999373 \\ 0.000877 \\ 0.031823 \\ 0.083111 \end{bmatrix}, \qquad f_2 = f(0.1) = f + 0.1u = \begin{bmatrix} 0.401937 \\ 4.000088 \\ 0.023182 \\ -0.041689 \end{bmatrix}.$$

From this we derive that

$$\mathrm{Diss}_{f_2}(g) \geq \mathrm{Diss}_f(g) + \frac{\gamma^2}{\lambda}\|f\|^2 \geq \mathrm{Diss}_f(g) + 32499\,\gamma^2.$$

Hence, for $\gamma = 0.1$ the dissipation becomes about at least 341. Thus we conclude that a perturbation of the design current of only 10% may lead to an increase of the dissipation by as much as a factor 21.

It is clear from the above example that small eigenvalues of the matrix $A(g)$ may give rise to instability of the network: a small perturbation of the design current may cause a large increase of the dissipation. We may conclude that for a stable network the eigenvalues of the matrix $A(g)$ should be bounded well away from zero.

In the sequel we will consider $1/\lambda_{\min}(A(g))$ as a measure for the stability of the network. For the above example this quantity is about 2031.

Shor's lemma

In the sequel we need the following lemma. In this lemma the notation $M \succeq 0$ represents a so-called matrix inequality; it means that M is s positive semidefinite matrix.

Lemma 1 *A quadratic form* $x^T Ax + 2b^T x + c$ *is* ≥ 0 *for all* x *if and only if*

$$\begin{pmatrix} A & b \\ b^T & c \end{pmatrix} \succeq 0 \text{ or, equivalently } \begin{pmatrix} A & -b \\ -b^T & c \end{pmatrix} \succeq 0.$$

Proof: The proof consists of a sequence of logically equivalent statements, as follows:

$$\begin{aligned}
\forall x \ &: \ x^T Ax + 2b^T x + c \geq 0 \ \Leftrightarrow \\
\forall (t \neq 0, x) \ &: \ t^{-2} x^T Ax + 2t^{-1} b^T x + c \geq 0 \ \Leftrightarrow \\
\forall (t \neq 0, x) \ &: \ x^T Ax + 2tb^T x + ct^2 \geq 0 \ \Leftrightarrow \\
\forall (t, x) \ &: \ x^T Ax + 2tb^T x + ct^2 \geq 0 \ \Leftrightarrow \\
\forall (t, x) \ &: \ \begin{pmatrix} x \\ t \end{pmatrix}^T \begin{pmatrix} A & b \\ b^T & c \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \geq 0 \ \Leftrightarrow \ \begin{pmatrix} A & b \\ b^T & c \end{pmatrix} \succeq 0.
\end{aligned}$$

Variational Principle

Given a network g with external current f, and with the function $E_{g,f}(y)$ defined as before, we have:

$$E_{g,f}(y) = 2f^T y - y^T A(g) y.$$

Since $g \geq 0$, the matrix $A(g)$ is positive semidefinite, and hence $E_{g,f}(y)$ is a concave function. As a consequence, $E_{g,f}(y)$ is maximal if and only y is such that

$$A(g)\, y = f.$$

Note that this is exactly the equation for equilibrium. Thus we obtain the following Variational Principle:

> The potential y of a network g under an external current f is a maximizer of the quadratic form $E_{g,f}(y)$.

Note that in equilibrium, the (maximal) value of the function $E_{g,f}(y)$ is given by $f^T y$. In other words,

$$\mathrm{Diss}_f(g) = \max_y E_{g,f}(y) = \max_y \left(2f^T y - y^T A(g) y\right).$$

Lemma 2 $\mathrm{Diss}_f(g) \le \tau$ *holds if and only if*

$$y^T A(g) y - 2f^T y + \tau \ge 0, \quad \forall y.$$

Proof: $\mathrm{Diss}_f(g) \le \tau$ is equivalent to $\max_{y \in \mathbf{R}^m} \left[2f^T y - y^T A(g) y\right] \le \tau$. Clearly, this holds if and only if $2f^T y - y^T A(g) y \le \tau$ for all $y \in \mathbf{R}^m$, i.e., if and only if the quadratic form

$$y^T A(g) y - 2f^T y + \tau$$

is nonnegative for all $y \in \mathbf{R}^n$. •

Theorem 1 $\mathrm{Diss}_f(g) \le \tau$ *holds if and only if*

$$\begin{pmatrix} \tau & f^T \\ f & A(g) \end{pmatrix} \succeq 0.$$

Proof: This immediately follows from Lemma 2 and Shor's Lemma. •

The semidefinite representation of the dissipation enables us to formulate the RNTD problem as a so-called semidefinite optimization problem:

$$\min_{\tau,\, g} \left\{ \tau \;:\; \begin{pmatrix} \tau & f^T \\ f & A(g) \end{pmatrix} \succeq 0,\;\; e^T g \le w,\, g \ge 0 \right\}.$$

In the multi-current case we assume that the set $\mathcal{F}$ of current scenarios is a finite set:

$$\mathcal{F} = \{f_1, \ldots, f_k\}.$$

A big advantage of this model is that it can be easily adapted to obtain a RNTD that can withstand the currents f_i in $\mathcal{F}$ (not acting at the same time) in the best possible way.

$$\min_{\tau, g} \left\{ \tau \;:\; \begin{pmatrix} \tau & f_j^T \\ f_j & A(g) \end{pmatrix} \succeq 0,\, j = 1, \ldots, k,\;\; e^T g \le w,\, g \ge 0 \right\}.$$

The linear matrix inequalities (LMI's) express that the dissipation of the network determined by g does not exceed τ, for each of the currents $f_1, \ldots, f_k$.

A crucial question is if we can solve these models efficiently!

General conic optimization problems

A general conic optimization problem is a problem in the conic form

$$\min_{x \in \mathbf{R}^n} \left\{ c^T x \ : \ Ax - b \in \mathcal{K} \right\},$$

where $\mathcal{K}$ is a closed convex pointed cone. Examples of such cones are

- the non-negative orthant $\mathbf{R}^m_+$ (Linear Inequality constraints);
- the Lorentz (or second order, or ice-cream) cone $\mathbf{L}^m$ (Conic Quadratic constraints);
- the semidefinite cone $\mathbf{S}^m_+$, i.e. the cone of positive semidefinite matrices of size $m \times m$ (Linear Matrix Inequality (LMI) constraints);
- a direct product of such cones.

In all these cases conic optimization problems can be solved efficiently by an interior-point method.

Example of a multi-current network

We turn back to the instable network considered before. This network was designed for the single current $f_1 = f$. We saw that when it is subjected to the current f_2 the dissipation increases with a factor 21.

We can now optimize the network with respect to both currents by solving the multi-current model for $k = 2$ and f_1 and f_2 as external currents. The resulting network has the same topology as the original network but with a different vector g. The dissipation of the new network with respect to f_1 is 17.845762 (in stead of 16.176484). So the dissipation in the new network, when loaded with f_1 is about 10% higher than the minimal dissipation. On the other hand, with respect to f_2 the dissipation is now 19.582459, an increase of only 21%. This is an enormous improvement. We gain a lot in terms of stability, at a small cost in terms of optimality.

However, the new network is not yet stable. If we replace the design current by

$$f_3 = \begin{bmatrix} 0.006981 \\ 4.000602 \\ 0.401186 \\ 0.071248 \end{bmatrix},$$

the perturbation is not more than 10%, and the dissipation becomes 52.942055, an increase with more than a factor 3 with respect to the dissipation for f_1.

We finally consider the so-called robust RNTD problem, where we assume that the set of external currents $\mathcal{F}$ is an ellipsoid:

$$\mathcal{F} = \left\{ Qz \; : \; z^T z \leq 1 \right\}, \quad Q \in \mathbf{M}^{m \times p}.$$

The matrix Q can be chosen such that the ellipsoid $\mathcal{F}$ contains all possible external currents that might occur. Since the set $\mathcal{F}$ is infinite, we meet a difficulty not present in the case of finite $\mathcal{F}$, namely that the objective now is to minimize

$$\mathsf{Diss}_{\mathcal{F}}(g) := \sup_{f \in \mathcal{F}} \mathsf{Diss}_f(g), \tag{1}$$

which is the supremum of infinitely many semidefinite representable (SDR) functions. Fortunately, it is easy to get a semidefinite representation for $\mathsf{Diss}_{\mathcal{F}}(g)$. In the next theorem I_p denotes the unit matrix of order p.

Theorem 2 *One has* $\mathsf{Diss}_f(g) \leq \tau$ *for each* $f \in \mathcal{F}$ *if and only if*

$$\begin{pmatrix} \tau I_p & Q^T \\ Q & A(g) \end{pmatrix} \succeq 0.$$

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Proof: With $\mathsf{Diss}_{\mathcal{F}}(g)$ as defined before, we may write

$$\begin{aligned}
\mathsf{Diss}_{\mathcal{F}}(g) \le \tau \quad & \Leftrightarrow \quad x^T A(g) x - 2(Qz)^T x + \tau \ge 0, \quad \forall x\, \forall (z\ :\ z^T z \le 1) \\
& \Leftrightarrow \quad x^T A(g) x - 2(Qz)^T x + \tau \ge 0, \quad \forall x\, \forall (z\ :\ z^T z = 1) \\
& \Leftrightarrow \quad x^T A(g) x - 2(Q\tfrac{z}{\|z\|})^T x + \tau \ge 0, \quad \forall x\, \forall z \ne 0 \\
& \Leftrightarrow \quad (\|z\|\, x)^T A(g) (\|z\|\, x) - 2(Qz)^T(\|z\|\, x) + \tau z^T z \ge 0,\ \forall x\, \forall z.
\end{aligned}$$

Replacing $\|z\|\, x$ by $-y$ we obtain

$$\begin{aligned}
\mathsf{Diss}_{\mathcal{F}}(g) \le \tau \quad & \Leftrightarrow \quad \tau z^T z + 2 z^T Q^T y + y^T A(g) y \ge 0, \quad \forall y\, \forall z \\
& \Leftrightarrow \quad \begin{pmatrix} z \\ y \end{pmatrix}^T \begin{pmatrix} \tau I_p & Q^T \\ Q & A(g) \end{pmatrix} \begin{pmatrix} z \\ y \end{pmatrix} \ge 0, \quad \forall y\, \forall z \\
& \Leftrightarrow \quad \begin{pmatrix} \tau I_p & Q^T \\ Q & A(g) \end{pmatrix} \succeq 0.
\end{aligned}$$

This proves the theorem. ●

$$\mathcal{F} = \left\{ f = Qz \;:\; z^T z \leq 1 \right\}, \quad Q \in \mathbf{M}^{m \times p}.$$

Theorem 2 enables us to model the robust RNTD problem as follows:

$$\min_{\tau, g} \left\{ \tau \;:\; \begin{pmatrix} \tau I_p & Q^T \\ Q & A(g) \end{pmatrix} \succeq 0, \quad e^T g \leq w,\ g \geq 0 \right\}.$$

This model finds the network which is best able to withstand *all* the external currents in the ellipsoidal set $\mathcal{F}$. Note that it does not tell us how to choose the matrix Q. But it is clear that we should choose Q in such a way that the ellipsoid $\mathcal{F}$ contains all currents that possibly may occur. Below we will show the results for

$$Q = [f_1 \;\; 0.3 \times \|f_1\| \times I_4] = \begin{bmatrix} 0.002 & 1.200109 & 0 & 0 & 0 \\ 4.000 & 0 & 1.20010911 & 0 & 0 \\ 0.020 & 0 & 0 & 1.200109 & 0 \\ -0.050 & 0 & 0 & 0 & 1.200109 \end{bmatrix},$$

and compare with them all previous results.

Summary of results

1	g	LO (1L)	SDO (2L)	SDO (3L)	SDO (Rob.)
2	g_{12}	0.000265	0.087292	0.042106	0.066618
3	g_{13}	0.984046	0.899470	0.869742	0.797107
4	g_{14}	0.003257	0.003818	0.062046	0.069907
5	g_{15}	0	0	0.006339	0.057117
6	g_{23}	0	0	0	0
7	g_{24}	0	0	0.010774	0
8	g_{25}	0.000232	0.003535	0.008993	0.000005
9	g_{34}	0	0	0	0
10	g_{35}	0.010484	0.004465	0	0.009200
11	g_{45}	0.001715	0.001421	0	0.000046
12	Design dissipation	16.176484	19.582459	21.155707	21.738929
13	dissipation w.r.t f_1	16.176484	17.845762	18.575612	19.849607
14	dissipation w.r.t f_2	344.414583	19.582459	21.155707	22.277243
15	dissipation w.r.t f_3	54.156705	52.942055	21.155707	22.358241
16	$\lambda_{\text{min}}^{-1}(A(g))$	2031	209	74	15

Concluding remarks

It is now well-known that CO is a powerful tool for the mathematical modelling of inherently nonlinear problems. One may check the references listed below to observe that, with the exception of a few, all relevant papers appeared in the last 10 years. Indeed, the subject thanks its existence to the development of efficient solution methods for CO problems in the last decade. Especially the possibility of modelling robustness of a design in a computationally tractable way opens the way to many new applications.

A. Ben-Tal and A. Nemirovski, illustrated the use of CO models for solving robust truss topology design problems by giving some convincing examples. In the present paper we apply the same approach to the inherently more simple case of the robust resistor network topology design problem. It is shown that by using a semidefinite model the robustness of the design can be significantly improved. It may be expected that the extension to more general networks, with inductances and capacitators is more or less straightforward. This might be the subject of future research.

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